ONLINE LINEAR DISCREPANCY OF PARTIALLY ORDERED SETS

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This article is dedicated to Professor Endre Szemerédi on the occasion of his 70^{th} birthday. Among his many remarkable contributions to combinatorial mathematics and theoretical computer science is a jewel for online problems for partially ordered sets: the fact that $h(h+1)/2$ antichains are required for an online antichain partition of a poset of height h.

ABSTRACT. The linear discrepancy of a poset $\mathbf{P}$ is the least k for which there is a linear extension L of $\mathbf{P}$ such that if x and y are incomparable in $\mathbf{P}$, then $|h_L(x) - h_L(y)| \leq k$, where $h_L(x)$ is the height of x in L. In this paper, we consider linear discrepancy in an online setting and devise an online algorithm that constructs a linear extension L of a poset P so that $|h_L(x) - h_L(y)| \leq 3k - 1$, when the linear discrepancy of P is k. This inequality is best possible, even for the class of interval orders. Furthermore, if the poset P is a semiorder, then the inequality is improved to $|h_L(x) - h_L(y)| \leq 2k$. Again, this result is best possible.

1. INTRODUCTION

The concept of linear discrepancy for finite partially ordered sets (posets) was introduced by Tanenbaum, Trenk, and Fishburn in [9] and represents an effort to model a notion of "fairness", i.e., incomparable elements should be placed close together in a linear extension to avoid an implicit comparison being made when one has much greater height than the other.

In this paper, we will consider linear discrepancy in an online setting. We will show that there is an online algorithm that will construct a linear extension L of a poset $\mathbf{P}$ so that any pair of incomparable points in $\mathbf{P}$ are at distance at most $3k - 1$ in L when the linear discrepancy of $\mathbf{P}$ is k. When the poset is a semiorder, this distance can be lowered to $2k$. Both results are best possible.

The remainder of this paper is organized as follows. In the next section, we develop essential notation and terminology and discuss briefly some related work that motivates our line of research. The proofs of lower bounds for the inequalities in our main theorems are given in Section 4, while the proofs for upper bounds are given in Section 5.

2. NOTATION, TERMINOLOGY AND BACKGROUND

A partially ordered set (poset) $\mathbf{P}$ is a pair (X, P) where X is a finite *ground set* and P is a reflexive, antisymmetric and transitive binary relation on X. Rather than write $(x, y) \in P$, it is more common to write $x \leq y$ in P, and we just write $x \leq y$ when the partial order P is clear from the context. Of course, we write $x < y$ in P when $x \leq y$ in P and $x \neq y$. Distinct points x and y are *comparable* in P when either $x < y$ in P or $y < x$ in P; else they are *incomparable*, in which case we write $x \| y$ in P.

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We let $D(x) = \{y \in X \mid y < x\}$ and call $D(x)$ the down-set of x. The up-set of x, $U(x)$, is defined dually. We let $\mathrm{Inc}(x) = \{y \in X \mid y \| x\}$ and define[1] $\Delta(\mathbf{P}) = \max_{x \in X} |\mathrm{Inc}(x)|$. Throughout this paper, we adopt the standard convention of denoting a poset by a single symbol, so we write for example, $x \in \mathbf{P}$, $x < y$ in $\mathbf{P}$ and $z \| w$ in $\mathbf{P}$.

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$, and let $\mathbf{n}$ denote a linear order on n points, typically labeled as $0 < 1 < 2 < \cdots < n-1$. If $\mathbf{P}$ and $\mathbf{Q}$ are posets on disjoint ground sets, $\mathbf{P} + \mathbf{Q}$ denotes the disjoint union of $\mathbf{P}$ and $\mathbf{Q}$. Also, when $\mathbf{P}$ does not contain a subposet which is isomorphic to $\mathbf{Q}$, we say $\mathbf{P}$ *excludes* $\mathbf{Q}$.

The reader may find it helpful to consult Trotter's monograph [10] and survey article [11] for additional background material on combinatorial problems for partially ordered sets.

2.1. **Interval Orders and Semiorders.**

A poset $\mathbf{P}$ is called an *interval order* when there exists a function I assigning to each element $x \in \mathbf{P}$ a closed interval $I(x) = [l(x), r(x)]$ of the real line $\mathbb{R}$ such that $x < y$ in P if and only if $r(x) < l(y)$ in $\mathbb{R}$. We call the family $\{[l(x), r(x)] : x \in \mathbf{P}\}$ of intervals an *interval representation* of $\mathbf{P}$. It is easy to see that when $\mathbf{P}$ is an interval order, it has an interval representation with distinct endpoints. It is a well-known result of Fishburn [2] that a poset is an interval order if and only if it excludes $\mathbf{2} + \mathbf{2}$.

A *semiorder* is an interval order having an interval representation in which all intervals have length 1. Scott and Suppes [8] showed that an interval order is a semiorder if and only if it excludes $\mathbf{3} + \mathbf{1}$.

2.2. **Linear Discrepancy.**

Let $\mathbf{P} = (X, P)$ be a poset. A linear order L on X is called a *linear extension* of $\mathbf{P}$ if $x < y$ in L whenever $x < y$ in P. When L is a linear extension of $\mathbf{P}$ and $x \in \mathbf{P}$, the quantity $|\{y \in \mathbf{P} \mid y \leq x \text{ in } L\}|$ is called the *height* of x in L and is denoted $h_L(x)$.

The *linear discrepancy* of a linear extension L of $\mathbf{P}$, denoted $\mathrm{ld}(\mathbf{P}, L)$, is the least non-negative integer k so that $|h_L(x) - h_L(y)| \leq k$ whenever $x \| y$ in P. Note that $\mathrm{ld}(\mathbf{P}, L) = 0$ if and only if $\mathbf{P}$ is a linear order. Now let $\mathscr{E}(\mathbf{P})$ denote the family of all linear extensions of $\mathbf{P}$. The *linear discrepancy* of a poset $\mathbf{P}$, denoted $\mathrm{ld}(\mathbf{P})$, is then defined by

$$\mathrm{ld}(\mathbf{P}) := \min\{\mathrm{ld}(\mathbf{P}, L) : L \in \mathscr{E}(\mathbf{P})\}$$

We note that the parameter $\mathrm{ld}(\mathbf{P}, L)$ is called the uncertainty of L in [9].

Fishburn, Tannenbaum and Trenk [3] showed that the linear discrepancy of a poset is equal to the bandwidth of its cocomparability graph. The same authors noted in [9] that it follows from the work of Kloks, Kratsch, and Müller [6] on bandwidth that determining whether the linear discrepancy of a poset $\mathbf{P}$ is at most k is NP-complete.

In spite of the fact that the linear discrepancy of a poset is difficult to compute, it is very easy to approximate. The following result (with different notation) is given in [6].

Theorem 1. *If* $\mathbf{P}$ *is a poset, then* $\mathrm{ld}(\mathbf{P}, L) \leq 3\,\mathrm{ld}(\mathbf{P})$ *for every linear extension* L *of* $\mathbf{P}$.

The inequality of Theorem 1 is tight for all $d \geq 1$. This can be seen by considering the following poset with $3d + 1$ points: let A, B, and C be chains with d points, $d - 1$ points, and d points, respectively, such that $a < b < c$ for all points $a \in A$, $b \in B$, and $c \in C$. Let x and y be the remaining points such that $x \| a$, $x \| b$, $x < c$, $y > a$, $y \| b$, and $y \| c$, for all $a \in A$,

[1] This notation is nonstandard. In other settings, $\Delta(\mathbf{P})$ denotes the maximum degree in the comparability graph of $\mathbf{P}$.

$b \in B$, and $c \in C$. The linear extension with $A < x < B < y < C$ has discrepancy d, whereas the linear extension with $x < A < B < C < y$ has discrepancy $3d$.

We note here some key properties of linear discrepancy that will prove useful later in this paper.

Lemma 2.

(1) *Linear discrepancy is monotonic, i.e., if* $\mathbf{P}$ *is a subposet of* $\mathbf{Q}$*, then* $\text{ld}(\mathbf{P}) \leq \text{ld}(\mathbf{Q})$.
(2) *If* $\mathbf{P}$ *is an n-element antichain, then* $\text{ld}(\mathbf{P}) = n-1$*, so* $\text{ld}(\mathbf{P}) \geq \text{width}(\mathbf{P}) - 1$.
(3) $\Delta(\mathbf{P})/2 \leq \text{ld}(\mathbf{P}) \leq 2\Delta(\mathbf{P}) - 2$.

The nontrivial upper bound in the third statement of Lemma 2 is proved in [7], and we note that it remains open to settle whether the upper bound here can be improved to $\lfloor (3\Delta(\mathbf{P}) - 1)/2 \rfloor$. Special cases have been resolved in [1, 4, 7, 9].

In discussions to follow, we say a linear extension L of $\mathbf{P}$ is *optimal* if $\text{ld}(\mathbf{P}, L) = \text{ld}(\mathbf{P})$. It is shown in [9] that $\text{ld}(\mathbf{P}) = \text{width}(\mathbf{P}) - 1$, when $\mathbf{P}$ is a semiorder. To see this, just take an interval representation $\{[l(x), l(x)+1] : x \in X\}$ for $\mathbf{P}$ in which all endpoints are distinct. Let L be the linear extension of $\mathbf{P}$ defined by setting $x < y$ in L whenever $l_x < l_y$ in $\mathbb{R}$.

In [4], Keller and Young show that for an interval order $\mathbf{P}$, $\text{ld}(\mathbf{P}) \leq \Delta(\mathbf{P})$ with equality if and only if $\mathbf{P}$ contains an antichain of size $\Delta(\mathbf{P}) + 1$. They also show that this bound is tight even for interval orders of width 2.

3. Online Linear Discrepancy

In this paper, we consider linear discrepancy in an online setting. A *Builder* constructs a poset $\mathbf{P}$ from a class $\mathscr{P}$ of posets, one point at a time, and an *Assigner* assembles a linear extension L of $\mathbf{P}$, one point at a time. Even though the poset $\mathbf{P}$ and the linear extension L change with time, we use a single symbol for each. When Builder expands the poset $\mathbf{P}$ by adding a new point x, he will list those points presented previously that are (1) less than x in $\mathbf{P}$, and (2) greater than x in $\mathbf{P}$. Assigner will then insert x into a legal position in the linear extension L she had previously constructed just before x entered.

Given a class $\mathscr{P}$ of posets and an integer $k \geq 1$, we will investigate strategies for Builder that will enable him to construct a poset $\mathbf{P}$ from $\mathscr{P}$ with $\text{ld}(\mathbf{P}) \leq k$ so that Assigner will be forced to assemble a linear extension L of P with $\text{ld}(\mathbf{P}, L)$ much larger than k. We will also study algorithms for Assigner that will enable her to assemble a linear extension L of $\mathbf{P}$ with $\text{ld}(\mathbf{P}, L)$ relatively close to $\text{ld}(\mathbf{P})$. Of course, the inequality of Theorem 1 looms large in our discussion.

We will consider two different ways in which Builder can construct interval orders and semiorders. One way is for Builder to construct the poset one point at a time, just by listing the comparabilities for the new point x. It is easy for Assigner to be assured that Builder stays within the appropriate class by appealing to their characterization in terms of forbidden subposets.

However, we will also discuss the situation where Builder constructs an interval order or a semiorder by providing an interval representation one interval at a time. In this setting, Builder provides a closed interval $[l(x), r(x)]$ (with $r(x) = 1 + l(x)$ when $\mathbf{P}$ is a semiorder) for the new point. As we will see, Assigner will find this additional information quite valuable in constructing a linear extension which has linear discrepancy close to the optimal value.

With this notation and terminology in place, we can now give a formal statement of our principal theorems.

Theorem 3. *Let k be a positive integer. There is an online algorithm $\mathscr{A}$ for Assigner so that:*

(1) *If Builder constructs an arbitrary poset* $\mathbf{P}$ *with* $\mathrm{ld}(\mathbf{P}) = k$ *and Assigner assembles a linear extension L using Algorithm $\mathscr{A}$, then* $\mathrm{ld}(\mathbf{P}, L) \leq 3k - 1$. *This inequality is best possible, even if Builder is required to construct an interval order.*
(2) *If Builder constructs an semiorder* $\mathbf{P}$ *with* $\mathrm{ld}(\mathbf{P}) = k$ *and Assigner assembles a linear extension L using Algorithm $\mathscr{A}$, then* $\mathrm{ld}(\mathbf{P}, L) \leq 2k$. *This inequality is best possible.*

We will also prove the following result when Builder provides an interval representation.

Theorem 4. *Let k be a positive integer. There is an online algorithm $\mathscr{L}$ for Assigner so that:*

(1) *If Builder constructs an interval order* $\mathbf{P}$ *with* $\mathrm{ld}(\mathbf{P}) = k$ *by providing an interval representation and Assigner assembles a linear extension L using Algorithm $\mathscr{L}$, then* $\mathrm{ld}(\mathbf{P}, L) \leq 2k$. *This inequality is best possible.*
(2) *If Builder constructs a semiorder* $\mathbf{P}$ *with* $\mathrm{ld}(\mathbf{P}) = k$ *by providing an interval representation and Assigner assembles a linear extension L using Algorithm $\mathscr{L}$, then L is optimal.*

Before proceeding to the proofs of these two theorems, we pause to note that there is a substantive difference in the outcome when Builder is required to give an interval representation online. On the other hand, it can be seen in [5] that for online chain partitioning (and online graph coloring), there is no distinction between the two versions.

4. Lower Bounds for Online Linear Discrepancy

In this section, we provide strategies for Builder which establish lower bounds for the inequalities in Theorems 3 and 4.

Lemma 5. *For each $k \geq 1$, Builder can construct an interval order* $\mathbf{P}$ *with* $\mathrm{ld}(\mathbf{P}) = k$ *so that Assigner will be forced to assemble a linear extension L with* $\mathrm{ld}(\mathbf{P}, L) \geq 3k - 1$.

Proof. Builder constructs a poset $\mathbf{P}$ as follows. First, he presents a $k+1$-element antichain. After Assigner has linearly ordered these $k+1$ elements, Builder labels the L-least point as x and the others as members of an k-element antichain A. He then presents another $k+1$-element antichain, with all elements of the new antichain larger than all elements of $A \cup \{x\}$. After assigner has extended L, Builder labels the L-largest element as z and the other elements as as members of a k-element antichain D.

Builder then presents two new elements u and y with

(1) $a < u < d$ and $a < y < d$ for all $a \in A$ and $d \in D$, and
(2) $x < u < z$, $x \| y$ and $z \| y$.

By symmetry (up to duality), we may assume Assigner makes $y > u$ in L.

Next, Builder inserts two antichains, B and C, of sizes $k-1$ and $k-2$, respectively, so that:

(1) $x, a < b < c < u$ in P, for all $b \in B$ and $c \in C$.
(2) $y \| b$ and $y \| c$, for all $b \in B$ and $c \in C$.

Assigner must then insert all elements of B as a block of $k-1$ consecutive elements immediately over the L-largest element of A. Assigner must also insert all elements of C as a block immediately over the highest element of B. It follows that $h_L(y) - h_L(x) = 3k - 1$.

On the other hand, the linear order: $A < x < B < y < C < u < z < D$ shows that $\mathrm{ld}(\mathbf{P}) \leq k$. Also, it is also easy to see that $\mathbf{P}$ is an interval order with representation

$$A = [1,1], \quad x = [1,2], \quad D = [7,7], \quad z = [6,7],$$
$$y = [2,6], \quad u = [5,5], \quad B = [3,3], \quad C = [4,4],$$

where $A = [1,1]$, for instance, denotes that each element of A is represented by a copy of the interval $[1,1]$. □

We note that the interval order $\mathbf{P}$ constructed in Lemma 5 is not a semiorder, since it contains a subposet isomorphic to $\mathbf{3}+\mathbf{1}$. In fact, it contains a subposet isomorphic to $\mathbf{4}+\mathbf{1}$. We do not know whether Assigner can be forced to construct a linear extension L with $\mathrm{ld}(\mathbf{P},L) = 3k-1$ if Builder is restricted to interval orders that exclude $\mathbf{4}+\mathbf{1}$. We have been able to show that within this class, Builder can force Assigner to assemble a linear extension L with $\mathrm{ld}(\mathbf{P},L) \geq 2k+(k-1)/2$, but do not know if this is tight.

Lemma 6. *For each $k \geq 1$, Builder can construct a semiorder $\mathbf{P}$ with $\mathrm{ld}(\mathbf{P}) = k$ so that Assigner will be forced to assemble a linear extension L with $\mathrm{ld}(\mathbf{P},L) \geq 2k$.*

Proof. Builder constructs a poset $\mathbf{P}$ as follows. First, he presents an antichain of size $k+1$. When Assigner has linearly ordered these $k+1$ elements, Builder labels the L-least element as x and the remaining elements as members of a k-element antichain A. He then presents a k-element antichain B with

(1) $a < b$ in $\mathbf{P}$, for all $a \in A$ and $b \in B$, and
(2) $x \| b$, for all b in B.

Let b be the L-highest element of B. It follows that $h_L(b) - h_L(x) = 2k$, but the linear order $A < x < B$ shows that $\mathrm{ld}(\mathbf{P}) \leq 2k$. Also, it is clear that $\mathbf{P}$ is a semiorder. □

Lemma 7. *For each $k \geq 1$, Builder can construct an interval representation of an interval order $\mathbf{P}$ with $\mathrm{ld}(\mathbf{P}) = k$ so that Assigner will be forced to assemble a linear extension L with $\mathrm{ld}(\mathbf{P},L) = 2k$.*

Proof. Builder will construct a poset which is isomorphic to $\mathbf{2k}+\mathbf{1}$, which has linear discrepancy k. He begins by presenting the intervals $[1,4k-1]$ and $[2k,2k]$. If Assigner sets $[1,4k-1] < [2k,2k]$ in L, then Builder presents $[2k+1,2k+1]$, $[2k+2,2k+2]$, $\ldots$, $[4k-1,4k-1]$. Assigner has no choice but to put these $2k-1$ intervals above $[2k,2k]$ and therefore constructs a linear extension L of (a copy of) $\mathbf{2k}+\mathbf{1}$ with $\mathrm{ld}(\mathbf{P},L) = 2k$.

Similarly, if Assigner makes $[2k,2k] < [1,4k-1]$ in L, Builder proceeds to present $[1,1]$, $[2,2]$, $\ldots$, $[2k-1,2k-1]$, and again Assigner is forced to assemble a linear extension L with $\mathrm{ld}(\mathbf{P},L) = 2k$. □

5. Upper Bounds for Online Linear Discrepancy

In this section, we provide algorithms for Assigner to use in assembling a linear extension of a poset constructed by Builder. We note that these algorithms do not need to know anything about the class from which $\mathbf{P}$ arises, and they do not need to know anything about the linear discrepancy of $\mathbf{P}$. We focus first on posets constructed one point at a time, and later consider the case where Builder provides an interval representation of an interval order (or semiorder).

However, in order to motivate the specifics of our algorithm, we pause briefly to give examples showing that other reasonable strategies are non-optimal, even for the class of semiorders.

5.1. **Naïve online linear discrepancy algorithms for semiorders.** Perhaps the simplest online linear discrepancy algorithm that Assigner might use works as follows. When presented with a new point x, there is always a non-empty set of positions where x could be legally inserted. These positions are always consecutive in the linear extension L. A reasonable strategy is to insert x as close to the middle of these allowable positions as possible, say rounding down when there is an odd number of them. We refer to this algorithm as $\mathscr{M}$, since it places x in the middle of its allowable range.

However, we claim that for each $k \geq 1$, Builder can construct a semiorder $\mathbf{P}$ with $\mathrm{ld}(\mathbf{P}) = k$ so that Assigner is forced to assemble a linear extension L with $\mathrm{ld}(\mathbf{P}, L) = 3k-1$, provided she uses algorithm $\mathscr{M}$. As in the proof of Lemma 5, Builder starts by presenting two $k+1$-element antichains with all points of one less than all points of the other. Builder labels the L-least element as x and the L-greatest element as z with the remaining elements belonging to $A \cup D$ so that $a < d$ in $\mathbf{P}$, for every $a \in A$ and $d \in D$.

Builder then presents an element y with

(1) $x \| y$ and $z \| y$ in $\mathbf{P}$, and
(2) $a < y < d$ in $\mathbf{P}$, for all $a \in A$ and $d \in D$.

Next, Builder presents the elements of an $k-1$-element antichain C with

(1) $x < c$, $y \| c$ and $z > c$ in $\mathbf{P}$, for all $c \in C$, and
(2) $a < c < d$ in $\mathbf{P}$, for all $a \in A$, $c \in C$ and $d \in D$.

Using Algorithm $\mathscr{M}$, Assigner will place all elements of C as a block of consecutive elements immediately under y. Assigner then presents an antichain B of size $k-1$ with

(1) $x \| b$, $y \| b$ and $b < z$ in $\mathbf{P}$, for all $b \in B$, and
(2) $a < b < c$ in $\mathbf{P}$, for all $a \in A$, $b \in B$ and $c \in C$.

Assigner must then insert all elements of B as a block in between the highest element of A and the lowest element of C. It follows that $h_L(y) - h_L(x) = 3k-1$. On the other hand, the linear order $A < x < B < y < C < z < D$ shows that $\mathrm{ld}(\mathbf{P}) \leq k$. Furthermore, $\mathbf{P}$ is easily seen to be a semiorder.

We now consider a second algorithm that seems intuitive yet fails to be optimal. When presented with a new point x, this algorithm inserts x into the position that minimizes the linear discrepancy of the resulting linear extension, breaking ties by placing x as low as possible. Since this algorithm is in some sense greedy in its operation, we denote it by $\mathscr{G}$. Although we do not include the proof, it is straightforward to modify the argument for Algorithm $\mathscr{M}$ to verify the following claim: Builder can construct a semiorder $\mathbf{P}$, with $\mathrm{ld}(\mathbf{P}) = k$, that will force an Assigner using the $\mathscr{G}$ algorithm to order points as $x < A < B' < y < B'' < C < D < w$, where $B' \cup B'' = B$ and $|B'| = \lfloor |B|/2 \rfloor$. This linear extension has linear discrepancy $\lceil (k-1)/2 \rceil + 2k$.

We comment that it is not difficult to construct examples showing that $\mathscr{M}$, when applied to general posets, can be forced to construct linear extensions with linear discrepancy $3k$ (rather than $3k-1$) for posets of linear discrepancy k.

5.2. **An optimal online linear discrepancy algorithm.** Let $\mathbf{P}$ be a poset and let (x,y) be an ordered pair of elements from $\mathbf{P}$. We call (x,y) a *critical pair* if (1) $x \| y$ in $\mathbf{P}$; (2) $D(x) \subseteq D(y)$ in $\mathbf{P}$; and (3) $U(y) \subseteq U(x)$ in $\mathbf{P}$. If (x,y) is a critical pair and (y,x) is not a critical pair, then we call (x,y) a *one-way critical pair*. A linear extension L of a poset $\mathbf{P}$ is said to *reverse* a critical pair (x,y) when $y < x$ in L. The concept of critical pairs first surfaced in dimension theory, as it is easy to see that the dimension of poset $\mathbf{P}$ is the least positive integer t for which there exists a family $\{L_1, L_2, \ldots, L_t\}$ of linear extensions so that for every critical pair (x,y) in $\mathbf{P}$, there is some i for which (x,y) is reversed in L_i. The reader can find much more information on the role played by linear extensions in reversing critical pairs in [10].

By contrast, linear discrepancy is all about *preserving* critical pairs, as the following elementary but important proposition (see [4]) prevails:

Proposition 8. *Let* $\mathbf{P}$ *be a poset which is not a total order.*

(1) *If* L *is a linear extension of* $\mathbf{P}$ *and* x *and* y *are incomparable points with* $h_L(y) - h_L(x) = \mathrm{ld}(\mathbf{P}, L)$*, then* (x,y) *is a critical pair in* $\mathbf{P}$.
(2) *There exists an optimal linear extension* L *of* $\mathbf{P}$ *so that if* (x,y) *is a critical pair reversed by* L*, then* (y,x) *is also a critical pair.*

Accordingly, the online algorithm $\mathscr{A}$ we will define here endeavors to construct a linear extension that reverses few critical pairs. However, we note that in discussing critical pairs in an online setting, it may happen that a pair (x,y) is critical at one moment in time but is no longer critical at a later moment in time. The converse statement cannot hold.

Suppose a new point x enters the poset. Assigner considers the one-way critical pairs of the form (x,u), where u has already entered, as well as the one-way critical pairs of the form (v,x), where again v has already entered. If there are no one-way critical pairs of either type, then x is inserted in any legal position.

Suppose there are only one-way critical pairs of the form (x,u) but none of the other type. Let let u_0 be the lowest element of L for which (x,u_0) is a one-way critical pair. Insert x in any legal position which is under u_0. There is such a position, since $D(x) \subseteq D(u_0)$. Dually, if there are only one-way critical pairs of the form (v,x) but none of the other type, let v_0 be the highest element of L for which (v_0,x) is a one-way critical pair. Insert x in any legal position over v_0.

We are left to consider the case where there are one-way critical pairs of both types. Now let u_0 and v_0 be defined as above. Insert x in any legal position between u_0 and v_0. In making this statement, we note that u_0 and v_0 can occur in either order in L. If $v_0 < u_0$ in L, then some positions between u_0 and v_0 may be illegal, but there is at least one position between them which is legal. On the other hand, if $u_0 < v_0$ in L, then all positions between them are legal.

To analyze the behavior of $\mathscr{A}$, we require the following elementary proposition.

Proposition 9. *If* (x,y) *and* (y,z) *are critical pairs in a poset* $\mathbf{P}$*, then either* $x < z$ *in* $\mathbf{P}$ *or* (x,z) *is a critical pair in* $\mathbf{P}$.

Next we establish a key lemma concerning how $\mathscr{A}$ handles a configuration we denote by C. This configuration consists of four points x,y,z,w. Among these points, we require

(1) $z > w$, $z > x$, $y > w$, and $x \| y$ in $\mathbf{P}$, and
(2) (y,z) and (w,x) are critical pairs in $\mathbf{P}$.

Note that conditions (1) and (2) imply that (y,z) and (w,x) are in fact one-way critical pairs.

Lemma 10. *Algorithm* $\mathscr{A}$ *never constructs a linear extension* L *such that points forming a copy of* C *are ordered as* $x < w < z < y$ *in* L*, all points less than* x *in* L *are less than* y *in* $\mathbf{P}$*, and all points greater than* y *in* L *are greater than* x *in* $\mathbf{P}$.

Proof. We argue by contradiction. Consider the first time a copy of C is placed according to the conditions of the lemma. Notice that the last point presented must be one of x, y, w, or z, as the relationships amongst only those points induce C. By duality, it suffices to consider only the cases where x or w is the last point to enter the poset.

We first consider the possibility that x was the last point presented. Since $x < w$ in L and (w,x) is a critical pair, there is a point x' such that (x,x') is a critical pair and $x' < x$ in

L. Since (x,x') is a critical pair, we must have that $x' \| y$. This contradicts the fact that x, y, w, and z are placed according to the conditions of the lemma.

Now suppose w was the last point presented. Since $x < w$ in L and (w,x) is a critical pair, there is a point w' such that (w',w) is a critical pair but $w < w'$ in L. Now (w',x) is also a critical pair. Since (w',w) is a critical pair, we must have $w' < z$ and $w' < y$. Hence, $\{x,y,z,w'\}$ forms a copy of C placed in the forbidden order at an earlier stage, a contradiction. □

With Lemma 10 in hand, we are in position to analyze the performance of algorithm $\mathscr{A}$. We first consider the case of arbitrary posets in the following lemma.

Lemma 11. *Let $k \geq 1$. If Builder constructs a poset $\mathbf{P}$ with $\mathrm{ld}(\mathbf{P}) = k$, and Assigner uses Algorithm $\mathscr{A}$, she will assemble a linear extension L with $\mathrm{ld}(\mathbf{P},L) \leq 3k-1$.*

Proof. Let L be the linear extension of $\mathbf{P}$ assembled by Assigner. We show that $\mathrm{ld}(\mathbf{P},L) \leq 3k-1$. To the contrary, suppose that $\mathrm{ld}(\mathbf{P},L) = 3k$. Consider the first moment in time where there are points x and y with $x \| y$ and $h_L(y) - h_L(x) = 3k$. Let $S = \{s \mid x < s < y \text{ in } L\}$ and note that $|S| = 3k-1$. Also note that (x,y) is a critical pair in $\mathbf{P}$.

Let M be an optimal linear extension of $\mathbf{P}$. In view of Proposition 8, we may assume $x < y$ in M. Now let $Z = \{z \in S \mid y < z \text{ in } M\}$ and $W = \{w \in S \mid w < x \text{ in } M\}$. It follows that $z \| y$ for all $z \in Z$, so $|Z| \leq k$. Similarly, $|W| \leq k$. On the other hand, since $h_M(y) - h_M(x) \leq k$, we know $|S - (Z \cup W)| \leq k-1$. Thus $|Z| = |W| = k$ and $|S - (Z \cup W)| = k-1$. Now let z be the M-largest element of Z and let w be the M-least element of W. It follows that $h_M(x) - h_M(w) \geq k$, but since M is optimal, we know $h_M(x) - h_M(w) = k$ and thus (w,x) is a critical pair in $\mathbf{P}$. Similarly, (y,z) is a critical pair in $\mathbf{P}$. Also $z > w$ in $\mathbf{P}$. It follows that x, w, z, and y form a configuration C which L orders as $x < w < z < y$. Furthermore, all points less than x in L are less than y in $\mathbf{P}$ and all points greater than y in L are greater than x in $\mathbf{P}$ since $h_L(y) - h_L(x) = \mathrm{ld}(\mathbf{P},L)$. This contradicts Lemma 10. □

Lemma 12. *Let $k \geq 1$. If Builder constructs a semiorder $\mathbf{P}$ with $\mathrm{ld}(\mathbf{P}) = k$, and Assigner uses Algorithm $\mathscr{A}$, she will assemble a linear extension L with $\mathrm{ld}(\mathbf{P},L) \leq 2k$.*

Proof. Let L be the linear extension of $\mathbf{P}$ assembled by Assigner. We show that $\mathrm{ld}(\mathbf{P},L) \leq 2k$. To the contrary, suppose that $\mathrm{ld}(\mathbf{P},L) \geq 2k+1$. Consider the first moment in time where there are points x and y with $x \| y$ and $h_L(y) - h_L(x) \geq 2k+1$. Let $S = \{s \mid x < s < y \text{ in } L\}$ and note that $|S| = 2k$. Also note that (x,y) is a critical pair in $\mathbf{P}$.

Let M be an optimal linear extension of $\mathbf{P}$. In view of Proposition 8, we may assume $x < y$ in M. Now let $Z = \{z \in S \mid x < z \text{ in } \mathbf{P}\}$ and $W = \{w \in S \mid w < y \text{ in } \mathbf{P}\}$. If $Z = \emptyset$, then $x \| s$ for every $s \in S$, and since $x \| y$, we know that $\Delta(\mathbf{P}) \geq 2k+1$ which would imply that $\mathrm{ld}(\mathbf{P}) \geq k+1$. The contradiction forces Z to be nonempty. Similarly, $W \neq \emptyset$.

Choose $z \in Z$ and $w \in W$. Since $x \| y$, we must have $z \| y$ and $w \| x$ in $\mathbf{P}$. Since $\mathbf{P}$ is a semiorder and a semiorder is a special case of an interval order, we must have $z > w$ in $\mathbf{P}$. Observe that in a semiorder, whenever we have two distinct incomparable points u and v, then (at least) one of (u,v) and (v,u) is a critical pair. However, this implies that both of (y,z) and (w,x) are critical pairs in $\mathbf{P}$, while neither of (z,y) nor (x,w) is a critical pair. It follows that x, w, z, and y form the configuration C in the order forbidden by Lemma 10, a contradiction. □

5.3. **Online interval representations.** We now turn our attention to the situation where Builder constructs an interval order (or semiorder) by providing an interval representation, one interval at a time. Now Assigner will use the following algorithm, which we denote $\mathscr{L}$. A new point x comes with an interval $[l(x), r(x)]$ and this interval is fixed in time.

Assigner will then insert x into L so that elements are ordered by the left endpoints of their intervals, i.e., $u < v$ in L whenever $l(u) < l(v)$ in $\mathbb{R}$. Ties can be broken arbitrarily.

Lemma 13. *Let $k \geq 1$. If Builder constructs an interval order $\mathbf{P}$ by providing an interval representation, with $\mathrm{ld}(\mathbf{P}) = k$, and Assigner uses Algorithm $\mathscr{L}$, she will assemble a linear extension L with $\mathrm{ld}(\mathbf{P}, L) \leq 2k$.*

Proof. Let L be the linear extension of $\mathbf{P}$ assembled by Assigner. We show that $\mathrm{ld}(\mathbf{P}, L) \leq 2k$. To the contrary, suppose that $\mathrm{ld}(\mathbf{P}, L) \geq 2k + 1$. Clearly, we may stop when L first violates the conclusion. So we may choose points x and y so that $h_L(y) - h_L(x) = 2k + 1$.

Let $S = \{s \mid x < s < y \text{ in } L\}$. If $S \subseteq \mathrm{Inc}(x)$, then it follows that $\mathrm{ld}(\mathbf{P}) \geq \lceil \Delta(\mathbf{P})/2 \rceil \geq k + 1$. So there is a point $z \in S$ with $x < z$ in $\mathbf{P}$. It follows that $l(x) \leq r(x) < l(z) \leq l(y)$ in $\mathbb{R}$, which is a contradiction, since it implies that $x < y$ in $\mathbf{P}$. □

We state the analogous result for semiorders, noting that there is nothing to prove, as we are simply restating the well-known characterization of optimal linear extensions of semiorders.

Lemma 14. *Let $k \geq 1$. If Builder constructs a semiorder $\mathbf{P}$ by providing an interval representation, with $\mathrm{ld}(\mathbf{P}) = k$ and Assigner uses Algorithm $\mathscr{L}$, then she will assemble an optimal linear extension L.*

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