Degree Bounds for Linear Discrepancy of Interval Orders and Disconnected Posets

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Abstract

Let P be a poset in which each point is incomparable to at most Δ others. Tanenbaum, Trenk, and Fishburn asked in a 2001 paper if the linear discrepancy of such a poset is bounded above by $\lfloor (3\Delta-1)/2 \rfloor$. This paper answers their question in the affirmative for two classes of posets. The first class is the interval orders, which are shown to have linear discrepancy at most Δ, with equality precisely for interval orders containing an antichain of size $\Delta+1$. The stronger bound is tight even for interval orders of width 2. The second class of posets considered is the disconnected posets, which have linear discrepancy at most $\lfloor (3\Delta-1)/2 \rfloor$. This paper also contains lemmas on the role of critical pairs in linear discrepancy as well as a theorem establishing that every poset contains a point whose removal decreases the linear discrepancy by at most 1.

Key words:
linear discrepancy, interval order, poset, bandwidth, interval graph
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1. Introduction

Tanenbaum, Trenk, and Fishburn introduced the concept of linear discrepancy in [15] as a way of measuring how far apart linear extensions must place incomparable elements. They closed that paper with eight questions and challenges. One called for special results for linear discrepancy of interval orders, and another asked a question about bounding the linear discrepancy of a poset in terms of the maximum number of elements with which any element is incomparable. This paper provides results addressing these questions.

We begin by introducing some terminology and notation. If x and y are incomparable elements of a poset P, we will write $x \parallel_P y$ or simply $x \parallel y$. The set of all elements of P incomparable to x will be denoted $\mathrm{Inc}(x)$. To illustrate the relationship between a poset P and its co-comparability graph, we will define $\Delta(P) = \max_{x\in P} |\mathrm{Inc}(x)|$, which is the maximum degree in the co-comparability graph of P. When it is clear which poset is under consideration we will simply use Δ for $\Delta(P)$.

An *interval order* is a poset P for which we can associate a closed, bounded real interval $[\ell(x), r(x)]$ to each element $x \in P$ such that for all $x, y \in P$, $x <_P y$ if and only if $r(x) < \ell(y)$.

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The associated collection of intervals is called an *interval representation* of P. Without loss of generality, all interval orders in this paper are presented via an interval representation in which all endpoints are distinct. (Since we only consider finite interval orders, this is possible by simply adjusting duplicated endpoints by a very small amount.) An *interval graph* is the co-comparability graph of an interval order. If P and Q are disjoint posets, we denote by $P+Q$ the disjoint union of P and Q. When n is a positive integer, $\mathbf{n}$ will represent the natural linear order (or chain) on $\{1,2,\ldots,n\}$.

A *linear extension* L of a poset P is a linear order on the elements of P such that if $x <_P y$, then $x <_L y$. The height of an element x in a linear extension L will be denoted $h_L(x)$. We denote the *down-set* of x, $\{y \in P \mid y < x\}$, by $D(x)$. The *up-set* $U(x)$ is defined dually. A pair (x,y) is a *critical pair* in P if $x \|_P y$, $D(x) \subseteq D(y)$, and $U(y) \subseteq U(x)$. A linear extension L reverses a critical pair (x,y) if $y <_L x$. For any unfamiliar poset terminology or notation we refer the reader to Trotter's monograph [16]. More information on interval orders and interval graphs can be found in Fishburn's monograph [5].

For a poset P and a linear extension L of P, we denote the *linear discrepancy of* L as $\mathrm{ld}(P,L)$, and define $\mathrm{ld}(P,L) = \max_{x\|_P y} |h_L(x) - h_L(y)|$. The *linear discrepancy of* P, denoted $\mathrm{ld}(P)$, is the minimum of $\mathrm{ld}(P,L)$ over all linear extensions L. A related concept for a graph G on n vertices is *bandwidth*, which is the least integer k such that there exists a labelling of the vertices by $\{1,2,\ldots,n\}$ so that the labels of adjacent vertices differ by at most k.

In general, calculating the bandwidth of a graph is NP-hard, even for trees with maximum degree 3, as shown in [8]. In fact, Blache, Karpinski, and Wirtgen showed in [1] that even for trees there is no polynomial time approximation scheme for calculating the bandwidth. In contrast, Kleitman and Vohra showed in [12] that the bandwidth of an interval graph can be determined in polynomial time. The difficulties in calculating the bandwidth in the general case have led to a host of work on bounding and calculating the bandwidth for specific classes of graphs. (See [3, 13] for a survey of such results.)

To connect linear discrepancy and bandwidth, Fishburn, Tanenbaum, and Trenk showed in [6] that the linear discrepancy of a poset is equal to the bandwidth of its co-comparability graph. In [15], the same authors noted that this implies that determining if $\mathrm{ld}(P) \leq k$ is NP-complete. They also proved that the poset consisting of disjoint chains of sizes $a_1,\ldots,a_m$ has linear discrepancy $\sum_{i=1}^m a_i - 1 - \max_i \lfloor a_i/2 \rfloor$. As a special case of this formula, they noted $\mathrm{ld}(\mathbf{t}+\mathbf{t}) = \lfloor (3t-1)/2 \rfloor$ and asked if $\mathrm{ld}(P) \leq \lfloor (3\Delta(P)-1)/2 \rfloor$ for every poset P.

Trivially, $\mathrm{ld}(P) \leq 2\Delta(P) - 1$. The only general improvement upon this bound was provided by Rautenbach in [14]. Via observations regarding linear extensions, he showed that for a co-comparability graph G, $\mathrm{bw}(G) \leq 2\Delta(G) - 2$. Thus for P a poset, $\mathrm{ld}(P) \leq 2\Delta(P) - 2$. In [4], Choi and West improved this bound to $\lfloor (3\Delta(P)-1)/2 \rfloor$ for posets of width 2.

In this paper we strengthen Rautenbach's result for special classes of posets by proving two degree-based bounds on linear discrepancy. We first prove Theorem 1, a Brooks'-type Theorem [2] for the linear discrepancy of interval orders. This theorem establishes that an interval order P has linear discrepancy at most $\Delta(P)$, with equality if and only if P contains an antichain of size $\Delta(P)+1$. To show the tightness of the stronger bound, we present, for each r, an infinite family of interval orders having width 2, $\Delta(P) = r$, and $\mathrm{ld}(P) = r-1$. In order to facilitate computing the linear discrepancy of our family of examples, we prove two lemmas about the role of critical pairs in determining linear discrepancy. As a precursor to showing that $\mathrm{ld}(P) \leq \lfloor (3\Delta(P)-1)/2 \rfloor$ if P is a disconnected poset, we show that every poset contains a point whose removal decreases the linear discrepancy by at most one. The final section suggests avenues for future research.

2. Degree Bounds for Interval Orders

We note that it is implicit in the work of Fomin and Golovach [7] (via a pathwidth argument), that the bandwidth of an interval graph G is at most $\Delta(G)$, and therefore the linear discrepancy of an interval order P is at most $\Delta(P)$. However, there is a straightforward proof of this fact. Let L be the linear extension of P ordering the points according to right endpoint. If $x \parallel y$ with $r(x) < r(y)$, then, since $\ell(y) < r(x) < r(z)$ for all z between x and y in L, any element placed between x and y in L must be incomparable to y. Thus, there are at most $\Delta - 1$ elements between x and y and hence $\Delta(P) \geq \mathrm{ld}(P,L) \geq \mathrm{ld}(P)$. If $\mathrm{width}(P) = \Delta + 1$, then since $\mathrm{ld}(P) \geq \mathrm{width}(P) - 1$ as shown in [15], $\mathrm{ld}(P) \geq \Delta + 1 - 1 = \Delta$. Thus, $\mathrm{ld}(P) = \Delta$. Theorem 1 shows that if this is not the case, we can strengthen the upper bound.

Theorem 1. *An interval order P has linear discrepancy at most $\Delta(P)$, with equality if and only if it contains an antichain of size $\Delta(P) + 1$.*

Proof. By the previous remarks, we may assume P is an interval order that does not contain an antichain of size $\Delta + 1$. By induction, we may assume that P does not split into disjoint sets D and U such that $d < u$ for all $d \in D$ and $u \in U$, as otherwise $\mathrm{ld}(P) = \max\{\mathrm{ld}(D), \mathrm{ld}(U)\}$. Fix an interval representation of P, and let m be the interval with largest left endpoint. We may assume that m also has the largest right endpoint. (Since m must be maximal, we may do this by extending the interval corresponding to m to the right.)

Form a linear extension L of P by ordering the intervals by right endpoint. Take $x \in P - \{m\}$. Now since P does not split, x overlaps an interval z with $r(z) > r(x)$. Therefore, $z >_L x$. Since $y <_P x$ implies $r(y) < \ell(x)$, elements of $\mathrm{Inc}(x)$ less than x in L precede x immediately as a consecutive block in L. Since for $x \neq m$ there are at most $\Delta - 1$ elements incomparable to x that can appear to its left, we see that $h_L(x) - h_L(y) \leq \Delta - 1$ for any $y \parallel x$ with $y <_L x$.

It only remains to address the interval m. First observe that the elements of $\mathrm{Inc}(m) \cup \{m\}$ are consecutive as above. Further, note that m is incomparable only to maximal elements by our choice of m. Since the maximal elements of P are an antichain and $\mathrm{width}(P) \leq \Delta$, m is incomparable to at most $\Delta - 1$ points. Thus $h_L(m) - h_L(z) \leq \Delta - 1$ for all z incomparable to m. Therefore, $\Delta - 1 \geq \mathrm{ld}(P,L) \geq \mathrm{ld}(P)$. □

As a consequence of the equivalence of linear discrepancy and bandwidth, we have the following analogous result for the bandwidth of interval graphs.

Theorem 2. *The bandwidth of an interval graph G is at most $\Delta(G)$, with equality if and only if it contains a clique of size $\Delta(G) + 1$.*

The stronger bound provided in Theorem 1 is tight, witnessed by the poset formed by adding one cover to an antichain on $\Delta + 1$ points (i.e., $\mathbf{2} + \mathbf{1} + \mathbf{1} + \cdots + \mathbf{1}$). However, in this case the tightness is a consequence of the trivial lower bound $\mathrm{ld}(P) \geq \mathrm{width}(P) - 1$. In order to show that this upper bound is nontrivial, we produce for each r an infinite family of interval orders P with width 2, $\Delta(P) = r$, and $\mathrm{ld}(P) = r - 1$. The following two lemmas restricting the class of linear extensions that need to be considered will be helpful in establishing the linear discrepancy of the constructed posets.

Lemma 3. *For any linear extension L of a poset P, the maximum distance in L between incomparable elements is achieved only at critical pairs.*

Proof. Suppose x and y are such that $x <_L y$ and achieve the maximum distance between incomparable elements in L. If $x' <_P x$, then $x' <_L x$. Therefore, by the maximality of (x,y), $x' <_P y$ and hence $D(x) \subseteq D(y)$. Similarly, $U(y) \subseteq U(x)$. Thus (x,y) is a critical pair. $\square$

If (x,y) is a critical pair, we say that (x,y) is *bicritical* if (y,x) is also a critical pair.

Lemma 4. *Let P be a poset. There exists a linear extension of P that is optimal with respect to linear discrepancy and reverses no critical pairs that are not bicritical.*

Proof. Consider a linear extension L of P that reverses at least one non-bicritical critical pair. Among all non-bicritical critical pairs that L reverses, take (x,y) so that $h_L(x) - h_L(y)$ is minimal. Since L is a linear extension, $D(y) <_L y <_L x <_L U(x)$. Hence, any element w with $y <_L w <_L x$ is incomparable to both x and y, since (x,y) is a critical pair. Thus, we may form a new linear extension L' from L simply by switching the positions of x and y. Furthermore, since (x,y) is a critical pair, if $w <_L y$ and $w \parallel y$, then $w \parallel x$. Similarly, if $v >_L x$ and $v \parallel x$, then $v \parallel y$. Thus, the distance between a pair of incomparable points in L' is no larger than it is in L, so $\mathrm{ld}(P,L') \leq \mathrm{ld}(P,L)$.

If switching the positions of x and y has introduced a new reversed critical pair (that is not bicritical), then one point of the critical pair must be x or y, and the other must lie between them in L (and thus in L'). Let this point be z. By symmetry, we may assume that (y,z) is a critical pair that is not bicritical. Now $D(y) \subseteq D(z)$ and $U(y) \supseteq U(z)$. Since (x,y) is a critical pair, $D(x) \subseteq D(y) \subseteq D(z)$ and $U(x) \supseteq U(y) \supseteq U(z)$. Since $x \parallel z$, (x,z) is also a critical pair. Furthermore, since neither (x,y) nor (y,z) is bicritical, (x,z) is not bicritical.

Now notice that if (y,z) is reversed in L', then (x,z) is reversed in L. Since $y <_L z <_L x$, we obtain $h_L(x) - h_L(z) < h_L(x) - h_L(y)$, contradicting our choice of (x,y). Thus, L' reverses fewer non-bicritical critical pairs than L and does not increase its linear discrepancy. Thus, we may take any optimal linear extension of P and use this process until arriving at an optimal linear extension that does not reverse any non-bicritical critical pairs. $\square$

Thus equipped, we will define a family of interval orders $\{\mathbf{F}_k^t\}_{k\geq 3}^{t\geq 1}$ and show that if $k > t$, then $\mathrm{ld}(\mathbf{F}_k^t) = \Delta(\mathbf{F}_k^t) - 1$. For each $t \geq 1$ and $k \geq 3$ define the elements of the interval order $\mathbf{F}_k^t$ as follows:

- For $0 \leq i \leq t-1$ and $0 \leq j \leq k-2$, the interval $[2jt + 2i, 2jt + 2i + 1]$ is the element a_i^j.
- For $0 \leq j \leq k$, the interval $\left[2(j-1)t - \frac{1}{2}, 2jt - \frac{1}{2}\right]$ is the element b_j.

Figure 1 illustrates the interval representation of a general $\mathbf{F}_k^t$, while Figure 2 shows $\mathbf{F}_4^3$. Note that $|\mathrm{Inc}(a_i^j)| = 1$ for all i, j, $\left|\mathrm{Inc}(b_j)\right| = t+2$ for $1 \leq j \leq k-1$, and $|\mathrm{Inc}(b_0)| = |\mathrm{Inc}(b_k)| = 1$, so $\Delta(\mathbf{F}_k^t) = t+2$. We also observe that $\mathrm{width}(\mathbf{F}_k^t) = 2$.

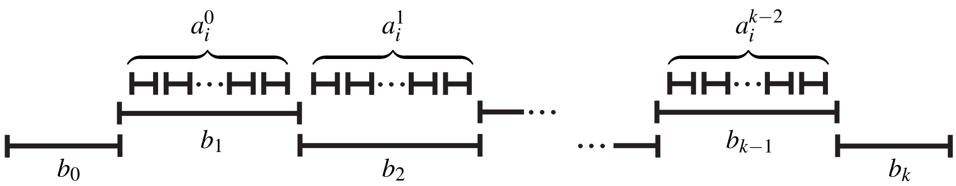

Figure 1: The interval order $\mathbf{F}_k^t$

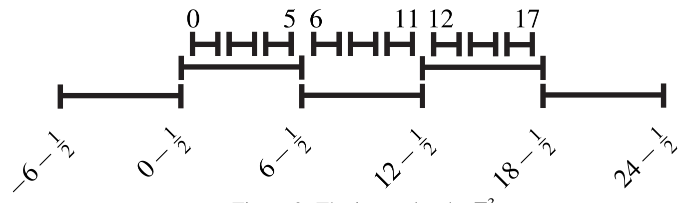

Figure 2: The interval order $\mathbf{F}_4^3$

Proposition 5. *The linear discrepancy of* $\mathbf{F}_k^t$ *is at least* $t+1-\lfloor t/k \rfloor = \Delta - 1 - \lfloor (\Delta-2)/k \rfloor$.

Proof. First we observe that the only critical pairs in $\mathbf{F}_k^t$ are of the form (b_i, b_{i+1}). Also, the remaining a_i^j points form a chain of height $t(k-1)$. By Lemma 4, to find a linear extension L that is optimal with respect to linear discrepancy, it suffices to consider only those having the property that L orders the b_i by index. Further, by Lemma 3 the distances between these pairs of points completely determine the linear discrepancy. Thus, we wish to distribute the $t(k-1)$ remaining points as equally as possible in the k gaps among the elements $\{b_0, b_1, \ldots, b_k\}$. This results in one gap containing at least $\lceil t(k-1)/k \rceil = t - \lfloor t/k \rfloor$ elements, implying $\mathrm{ld}\big(\mathbf{F}_k^t\big) \geq t+1-\lfloor t/k \rfloor$. □

For $k > t$, the upper bound from Theorem 1 and the lower bound from Proposition 5 combine to imply $\Delta - 1 \geq \mathrm{ld}\big(\mathbf{F}_k^t\big) \geq t+1 = \Delta - 1$. Hence, the stronger inequality of Theorem 1 is tight even for posets of width 2.

3. Linear Discrepancy and the Removal of Points

The *dimension* of a poset P is the least t such P is the intersection of some set of t linear orders. Dimension is a much studied property, as discussed in detail in [16]. There are some similarities between linear discrepancy and dimension, but at other times they stand in fairly stark contrast. For dimension, a theorem of Hiraguchi [9] guarantees that the removal of any point decreases a poset's dimension by at most one. However, removing the isolated point from $\mathbf{1}+\mathbf{n}$ illustrates that the removal of a single point can decrease the linear discrepancy of a poset by an arbitrarily large amount. Fortunately, there is always *some* element that behaves well.

Theorem 6. *For any poset there exists a point whose removal reduces the linear discrepancy by at most one.*

Proof. Let P be a poset. Suppose first that there are two minimal elements x and x' of P with the same up-set. Let L be a linear extension of $P - \{x'\}$ that is optimal with respect to linear discrepancy. Create a new linear extension L' by inserting x' immediately below x in L. It is clear that L' is a linear extension of P. Furthermore, since $\mathrm{Inc}(x) - \{x'\} = \mathrm{Inc}(x') - \{x\}$, the linear discrepancy of L' is at most one more than the linear discrepancy of L. Thus the removal of x decreases $\mathrm{ld}(P)$ by at most one.

If no two minimal elements have the same up-set, then there is a minimal element z such that there is no critical pair of the form (y,z). (A minimal element z with $|U(z)|$ maximum has this property.) Consider a linear extension L of $P - \{z\}$ that is optimal with respect to linear discrepancy. Let s be the element of $U(z) \cup \{v \mid (z,v) \text{ is a critical pair}\}$ for which $h_L(s)$ is minimal. Form a linear extension L' of P by inserting z immediately below s. By construction, L' is a linear extension of P. Since we only wish to show that $\mathrm{ld}(P,L')$ is at most one more than the linear discrepancy of $\operatorname{ld}(P-\{z\},L)$, the only obstructions are of the form $z \parallel z'$. By Lemma 3 and our choice of z, we may restrict our attention to critical pairs (z,z') with $h_{L'}(z')-h_{L'}(z)=\operatorname{ld}(P)$.

If $s \in U(z)$, our choice of s and z' imply that $s <_L z'$, and thus we must have $s \parallel z'$, as otherwise z and z' are comparable. If $s \notin U(z)$, then (z,s) is a critical pair, so $U(s) \subseteq U(z)$ and in particular $s \parallel z'$, as otherwise we would have $z' >_P z$. Now $\operatorname{ld}(P)=h_{L'}(z')-h_{L'}(z)=h_L(z')-h_L(s)+1 \leq \operatorname{ld}(P-\{z\})+1$. Hence the linear discrepancy of $P-\{z\}$ is at least $\operatorname{ld}(P)-1$ as desired. □

A poset P is *k-discrepancy irreducible* if $\operatorname{ld}(P)=k$ and $\operatorname{ld}(P-\{x\})<k$ for any $x \in P$. This concept has been used in [10, 11] to provide, together with the work of Tanenbaum, Trenk and Fishburn in [15], a complete forbidden subposet characterization of posets with linear discrepancy at most two. However, without Theorem 6, it is not immediate that linear discrepancy irreducibility is analogous to dimension irreducibility. Specifically, it was not known whether having linear discrepancy at least k assured the existence of a k-discrepancy irreducible subposet, except for $k \in \{1,2,3\}$ as shown in [11, 15]. However, as a consequence of Theorem 6 we have the following corollary.

Corollary 7. *If $\operatorname{ld}(P) \geq k$, then P contains a k-discrepancy irreducible subposet.*

4. Linear Discrepancy of Disconnected Posets

With Theorem 6 established, we are prepared to prove a second degree bound for linear discrepancy.

Theorem 8. *A disconnected poset P has linear discrepancy at most $\left\lfloor \frac{3\Delta(P)-1}{2} \right\rfloor$.*

Proof. We proceed by contradiction. Suppose P is a counterexample that is minimal in the number of elements, and hence irreducible with respect to linear discrepancy. If there is an isolated point $x \in P$, then $\operatorname{ld}(P) \leq |P|-1 = |\operatorname{Inc}(x)| = \Delta(P)$. Thus, P cannot be a counterexample. Therefore, P cannot have an isolated point. Hence, $P-\{x\}$ is disconnected for all $x \in P$. In particular, since $\Delta(P-\{x\}) \leq \Delta(P)$ for al $x \in P$, minimality and Theorem 6 imply $\operatorname{ld}(P) = \lfloor (3\Delta(P)-1)/2 \rfloor + 1$. Furthermore, Theorem 6 and the irreducibility of P guarantee the existence of a point x so that $Q = P-\{x\}$ has $\operatorname{ld}(Q)=\operatorname{ld}(P)-1$. Suppose that $\Delta(Q) \leq \Delta(P)-1$. By the minimality of P, the desired degree bound holds for Q, and therefore we have

$$\left\lfloor \frac{3\Delta(P)-1}{2} \right\rfloor = \operatorname{ld}(Q) \leq \left\lfloor \frac{3\Delta(Q)-1}{2} \right\rfloor \leq \left\lfloor \frac{3\Delta(P)-4}{2} \right\rfloor = \left\lfloor \frac{3\Delta(P)-2}{2} \right\rfloor - 1 < \left\lfloor \frac{3\Delta(P)-1}{2} \right\rfloor.$$

Hence, it follows that $\Delta(Q)=\Delta(P)$.

Since Q is disconnected, we may let (A,B) be a partition of Q witnessing this fact, named so that $|A| \leq |B|$. Observe that $\Delta(A) \leq \Delta(P)-|B|$ and $\Delta(B) \leq \Delta(P)-|A|$. Let L_B be an optimal linear extension of B, and let L_A be an arbitrary linear extension of A. Form a linear extension L of Q by taking the first $\lceil |B|/2 \rceil$ elements of L_B, then all the elements of A ordered by L_A, and finally the remaining elements of L_B. Now $\operatorname{ld}(Q) \leq \operatorname{ld}(Q,L)$, and so in particular,

$$\left\lfloor \frac{3\Delta(P)-1}{2} \right\rfloor \leq \max\left\{ |A| + \left\lceil \frac{|B|}{2} \right\rceil - 1, |A| + \operatorname{ld}(B) \right\}.$$

Suppose first that $\lfloor (3\Delta(P)-1)/2 \rfloor \leq |A| + \operatorname{ld}(B)$. Now $\operatorname{ld}(B) \leq 2\Delta(B)-2$ by Rautenbach's bound in [14]. Therefore, $\lfloor (3\Delta(P)-1)/2 \rfloor \leq |A| + 2\Delta(B) - 2$. Combining this with the observation that $\Delta(B) \leq \Delta(P)-|A|$, we obtain the bound $|A| \leq 2\Delta(P) - \lfloor (3\Delta(P)-1)/2 \rfloor - 2$. Since

$\text{ld}(Q) = \lfloor (3\Delta(P)-1)/2 \rfloor$, we have $|Q| = |A| + |B| \geq \lfloor (3\Delta(P)-1)/2 \rfloor + 1$. Therefore, $|B| \geq 2\lfloor (3\Delta(P)-1)/2 \rfloor + 3 - 2\Delta(P) \geq \Delta(P)+1$, a contradiction to the fact that $|B| \leq \Delta(Q) = \Delta(P)$.

Now we suppose $\lfloor (3\Delta(P)-1)/2 \rfloor \leq |A| + \lceil |B|/2 \rceil - 1$. Since $|A| \leq |B|$ and $|B| \leq \Delta(P) - \Delta(A)$, we then have

$$\left\lfloor \frac{3\Delta(P)-1}{2} \right\rfloor \leq |A| + \left\lceil \frac{|B|}{2} \right\rceil - 1 \leq \left\lceil \frac{3|B|-2}{2} \right\rceil \leq \left\lceil \frac{3\Delta(P)-3\Delta(A)-2}{2} \right\rceil .$$

Therefore, $\Delta(A) = 0$ and $|B| \leq \Delta(P)$. Similarly,

$$\left\lfloor \frac{3\Delta(P)-1}{2} \right\rfloor \leq \left\lceil \frac{2|A|+|B|-2}{2} \right\rceil \leq \left\lceil \frac{3\Delta(P)-2\Delta(B)-2}{2} \right\rceil = \left\lceil \frac{3\Delta(P)-2}{2} \right\rceil - \Delta(B).$$

Hence $\Delta(B) = 0$, and Q is the sum of two chains. By the formula for the linear discrepancy of the sum of chains, $\text{ld}(Q) = \lceil |B|/2 \rceil + |A| - 1$. Therefore $|A| = |B| = \Delta(P)$. In this situation, we see that we cannot form P from Q by the addition of a single point, since $\Delta(P) = \Delta(Q)$ and P is disconnected. Therefore, if P is a disconnected poset, $\text{ld}(P) \leq \lfloor (3\Delta(P)-1)/2 \rfloor$. □

5. Conclusions and Future Work

Theorem 8 is quite unusual in that there are few results stated only for disconnected posets. Adding a new element greater than all the elements in a poset yields a connected poset, and for most combinatorial questions, this does not change anything. However, the proof of Theorem 8 hinges on the large number of incomparabilities in disconnected posets. We see no reason to believe that the proposed bound $\text{ld}(P) \leq \lfloor (3\Delta(P)-1)/2 \rfloor$ does not hold in general, but at the same time we do not see how our methods could be extended.

Any improvement to the best known bound of $2\Delta(P)-2$, such as a result of the form $\text{ld}(P) \leq (2-\varepsilon)\Delta(P)$, would be very welcome. In fact, even the question of whether the proposed bound holds for posets with $\Delta(P) = 4$ (i.e., whether the correct upper bound is 5 or 6) is open. Perhaps an answer to this question would give additional insight into the larger problem.

Another intriguing direction for future work is to explore the relationship between linear discrepancy and dimension through their dependence on critical pairs. It is possible that the relationship is simply a fortunate coincidence. However, if there were an intuitive explanation for this relationship, it would perhaps suggest a proof of the conjecture that if $\text{ld}(P) = \dim(P) = n \geq 5$, then P contains the standard example $\mathbf{S}_n$ as a subposet. (See [15, 17].) Considering that the class of interval orders contains poset of arbitrary large dimension but does not contain the standard examples for $n > 1$, it would be interesting to see if it can be proved that for interval orders of dimension at least 5, $\text{ld}(P) \neq \dim P$.

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